JOURNAL OF APPROXIMATION THEORY 48, 303–321 (1986)

The Differentiability of Fourier Gap Series and "Riemann's Example" of a Continuous, Nondifferentiable Function

WOLFRAM LUTHER

Rheinisch-Westfälische Technische Hochschule Aachen, 51 Aachen, West Germany

Communicated by P. L. Butzer

Received May 11, 1984

DEDICATED TO THE MEMORY OF GÉZA FREUD

We give a general Tauberian gap theorem for a class of Fourier kernels which includes that of the Hankel transform $F(x)=\int_0^\infty \sqrt{xu}\, J_\nu(xu)\, f(u)\, du,\ \nu \geqslant -\frac{1}{2}$. Further, we discuss applications to Fourier gap series and the differentiability of $g(x)=\sum_{n=1}^\infty (\sin \pi n^2 x)/\pi n^\mu,\ 1 \leqslant \mu < 3$, a series supposedly due to Riemann, studied by G. H. Hardy in 1916.

1. INTRODUCTION

We are interested in the asymptotic behavior of the Fourier gap series

$$c(x)+s(x) := \sum_{k=0}^{\infty} (a_k \cos \lambda_k x + b_k \sin \lambda_k x),\ x \to x_0,$$

$$0=\lambda_0<\lambda_1<\lambda_2< \cdots <\lambda_n \to \infty, \qquad \lambda_{k+1}-\lambda_k \geqslant r_0 \lambda_k^\beta L_2(\lambda_k),\ 0<\beta \leqslant 1,$$

$$r_0>0,\ L_2 \text{ slowly varying},$$

and will show that most of the results concerning nondifferentiability of certain series are special cases of a general Tauberian gap remainder theorem. One such case is:

Suppose $\sum_{k=0}^\infty (|a_k|+|b_k|)<\infty$ and L_1, L_2 are slowly varying functions. Then the conditions

$$\left.\begin{matrix} c(0)-c(x) \\ s(x) \end{matrix}\right\} = O(x^m L_1(1/x)) \text{ [resp. } o(\cdots)] \text{ as } x \to 0+,\ m>0,$$

0021-9045/86 $3.00

imply that

$$\sum_{k=n}^{\infty} \begin{Bmatrix} a_k \\ b_k \end{Bmatrix} = O((u^{\beta} L_2(u))^{-m} L_1(u)) \text{ [resp. } o(\cdots)] \text{ as } u \to \infty.$$

We will show that differentiability depends strongly on the asymptotic behavior of the tail sums

$$\sum_{k=n}^{\infty} (a_k \cos \lambda_k x + b_k \sin \lambda_k x).$$

The particular case $\beta = 1$, $L_2 = 1$, i.e., the case of Hadamard gaps, was completely treated by Freud [4, 5], Hsieh T'ing-Fan [10], Belov [1] and the author [12]. It was indeed Professor Freud who in 1962 was the first to return to this old problem.

Smaller gaps are much more delicate to handle and often demand deep results from number theory. However, we can show that the series

$$\sum_{n=2}^{\infty} \frac{\sin(\pi n^{\nu} \log^{\rho} nx + \phi_n)}{\pi n^{\mu} \log^{\rho} n} \quad (1 \leqslant \mu \leqslant \nu - 1,\ \rho \in \mathbb{R};\ \mu = 1,\ \rho > 1;\ \phi_n \in \mathbb{R})$$

is nowhere differentiable. The best known example of this type is the so-called Riemann function

$$g_{\mu}(x) = \sum_{n=1}^{\infty} \frac{\sin \pi n^2 x}{\pi n^{\mu}}, \qquad 1 \leqslant \mu < 3.$$

Hardy [8] established the nondifferentiability of $g_{\mu}(x)$ for all irrational values of x (and some rational) and $\mu < 5/2$. Interest in this problem has been revived by contributions of Gerver [6, 7], Queffelec [18], Smith [20], Neuenschwander [14], Segal [19], Mohr [13], and Itatsu [11] as well as Butzer and Stark [2]. The basic result of these contributions is that g_2 has no finite derivative at any point other than those of the form $x = (2A+1)/(2B+1)$, where it has derivative $-1/2$.

In the second part of this paper we will deduce the following:

THEOREM 2. *The function* $g_{\mu}(x)$ *has a finite derivative at* $x = r/s$, $(r, s) = 1$, $0 \leqslant x \leqslant 1$, $3/2 < \mu < 3$, *if and only if* $rs \equiv 1 \pmod 2$. $g_{\mu}(x)$ *is nowhere differentiable if* $1 \leqslant \mu \leqslant 3/2$.

More precisely, $g_{\mu}(x)$ cannot satisfy the condition

$$g_{\mu}\left(\frac{r}{s} + h\right) - g_{\mu}\left(\frac{r}{s}\right) = o(|h|^{(\mu-1)/2}) \qquad \text{as} \quad h \to 0$$

for any rational $x = r/s$, $(r, s) = 1$, $rs \not\equiv 1 \pmod 2$.

2. General Tauberian Theorem

Let us formulate the general gap theorem needed in a convenient form and now list the assumptions to be used. All functions are assumed to be real and measurable. We consider the transform

$$F(x)=\int_0^\infty k(xu)\,f(u)\,du. \tag{2.1}$$

Assumptions on the function f:
There are constants $\alpha_1>\alpha\geqslant 0$ such that:

(a) $u^\alpha f(u)\in L^1(0,1)$,

(b) $f\in BV[1,\infty)$,

(c) $f(u)\to 0$ as $u\to\infty$,

(d) $f(u)=o(u^{-1-\alpha}),\ 0<u\leqslant 1.$ (2.2)

(b) together with (c) can be replaced by:

(b^*) $f\in L^2[1,\infty)$ and $\int_0^\infty k(xu)\,f(u)\,du$ converges for $0<x<a$,

(c*) f bounded in $[1,\infty)$.

Assumptions on the kernel k:

(C_1) $k(u)$ and $k_1(u):=\int_0^u k(x)\,dx$ are bounded in $0\leqslant u<\infty$,

(C_2) $k(u)=O(u^\alpha)$ as $u\to 0+$, $\alpha>0$ or, in the case $\alpha=0$, $k(u)=k(0)+O(u^{\alpha'})$ as $u\to 0+$, $k(0)\neq 0$, $\alpha'>0$, and $k_2(u):=\int_0^u k_1(x)\,x^{-1}\,dx$ is bounded in $0\leqslant u<\infty$.

$k^M(s)$ denotes the Mellin transform of $k(u)$:

$$k^M(s)=\int_0^\infty u^{s-1}k(u)\,du,$$

the integral being assumed to converge absolutely or conditionally. We need a further assumption:

(C_3) $k^M(s)\,k^M(1-s)=1,\ 0<\operatorname{Re} s<1.$

Then, k satisfying (C_1), (C_2), (C_3) is a symmetric Fourier kernel and (2.1) defines a unitary transformation on $L^2(0,\infty)$. The following condition (C_4) is satisfied by a large number of Fourier kernels, including that of the Hankel transform for which $k(u)=\sqrt{u}\,J_\nu(u)$, $\nu\geqslant -1/2$.

(C_4) $k^M(s)$ is meromorphic and has no zeros in $-\alpha_1 < \operatorname{Re} s < 1+\alpha$, $\alpha_1 > \alpha$, and there holds uniformly in the strip as $|\operatorname{Im} s| \to \infty$

$$1/k^M(s) = O((1+|s|)^{-\operatorname{Re} s + 1/2}).$$

We now introduce two classes of auxiliary functions: $F(x)$ will be dominated by $x^{m-1}\, L(1/x)\, w(x)$ $(x \to 0+)$, where w is positive and non-decreasing such that

$$w(xy) \leqslant w_0 y^b w(x), \qquad w_0 > 0,\ x > 0,\ y \geqslant 1,\ b \geqslant 0,$$

and L slowly varying in the sense of Karamata, i.e., L positive, measurable, locally bounded and

$$L(\lambda u)/L(u) \to 1 \text{ as } u \to \infty \qquad \text{for every } \lambda > 0.$$

The most important case is $w(x) = x^b$, $L(u) = |\log u|^\rho$, $\rho \in \mathbb{R}$.

We now come to our general Tauberian gap remainder theorem.

THEOREM 1. *If*

(i) *f and k satisfy the assumptions* (2.2) *and* (C_1), (C_2), (C_3), (C_4), *respectively,*

(ii) *for all $u \geqslant u_0$ there is an interval $I(u)$ with*

$$u \in I(u) \quad \text{and}\ |I(u)|/(u^\beta L_2(u)) \geqslant r_0 > 0, \quad 0 < \beta \leqslant 1,$$

such that $f(u)$ fullfils the following Tauberian condition in $I(u)$:

$$\sup_{\tilde{v}, v \in I(u), |\tilde{v} - v| \leqslant r_0 v^\beta L_2(v)} |f(\tilde{v}) - f(v)| \leqslant v^{-m\beta + b(1-\beta)} L_1(v)\, L_2^{-m-b}(v)\, w(1/v),$$

where $0 < m$ ($0 \leqslant m$ for $\alpha > 0$) and $0 < m + b < \alpha_1 + 1$,

(iii) *L_1 and L_2 are slowly varying, $u^{\beta-1} L_2(u)$ is nonincreasing and L_1 satisfies the inequality*

$$L_1(u^d)/L_1(u) \leqslant K_{d_0}, \qquad 0 < 1/d_0 \leqslant d \leqslant d_0,\ u \geqslant u_0,$$

for each finite d_0.

Then under the further hypothesis

$$F(x) = \int_0^\infty k(xu)\, f(u)\, du = O(x^{m-1} L_1(1/x)\, w(x)) \text{ as } x \to 0+,$$

it follows that

$$f(u) = O(u^{-m\beta + b(1-\beta)} L_1(u)\, L_2^{-m-b}(u)\, w(1/u)) \text{ as } u \to \infty.$$

Remark. The "O" result can be replaced by the corresponding "o" result.

The proof is to be found in [12, pp. 70–72]. However, we give an outline. First transform the integral (2.1) into the convolution form: $x \to e(-2\pi x)$, $u \to e(2\pi y)$, $e(x) = \exp(x)$, $\phi(x) = e(2\pi x)\, f(e(2\pi x))$, $K(x) = 2\pi k(e(-2\pi x))$, $\psi(x) = F(e(-2\pi x))$, $\psi(x) = (K \times \phi)(x) := \int_{-\infty}^{\infty} K(x-y)\, \phi(y)\, dy$. Then define a suitable function Q via $Q \times \psi(x) = (Q \times K) \times \phi(x)$ for all real x, and derive the important relation

$$|f(e(2\pi x))| \leqslant 4 \sup_{0 \leqslant v-y \leqslant 2/\Omega} (f(e(2\pi(x-y)))\, E(y) - f(e(2\pi(x-v)))\, E(v)) + 6e(-2\pi x)|Q \times \psi(x)| =: T_1 + T_2,$$

say, where $E(y) = e(-\xi\Omega^2 \cdot y^2)$, $\xi > 0$. Next one estimates the first term T_1 on the right side with the aid of the Tauberian condition

$$\sup_{\tilde{y}, y \in I(e(2\pi x)), |\tilde{y}-y| \leqslant r_2 e(2\pi(\beta-1)y) L_2(e(2\pi y))} |f(e(2\pi\tilde{y})) - f(e(2\pi y))|$$
$$= O(e(-2\pi\beta m y)\, e(2\pi b(1-\beta)\, y)\, L_2^{-m-b}(e(2\pi y))\, L_1(e(2\pi y))\, w(e(-2\pi y))),$$

where $|I(e(2\pi x))| \geqslant re(2\pi(\beta-1)\, x)\ L_2(e(2\pi x))$, $r > 0$, $x \geqslant x_0$. If $2/\Omega \leqslant \min(r_2, \delta/8)\ e(2\pi(\beta-1)\, x)\ L_2(e(2\pi x))$, and ξ is small enough, we get for some $r_1 < 1$

$$T_1 \leqslant K_1 \sup_{\delta e(2\pi(\beta-1)x) L_2(e(2\pi x)) \leqslant 8|v| \leqslant 4d} |f(e(2\pi(x-v)))\, e(-\xi\Omega^2 v^2/2)|$$
$$+ r_1|f(e(2\pi x))| + K_2 e(-\xi\Omega^2 d^2/16)$$
$$+ \tilde{K}_2 e(-2\pi\beta m x)\, e(2\pi b(1-\beta)\, x)$$
$$\times L_2^{-m-b}(e(2\pi x))\, L_1(e(2\pi x))\, w(e(-2\pi x)),$$

when $x \in X$, where X denotes the set of all $x \geqslant x_0$ whose distance from both ends of I is greater than $(\delta/4)\ e(2\pi(\beta-1)\, x)\ L_2(e(2\pi x))$, $\delta = \delta(r)$. For the second term T_2 we find

$$T_2 \leqslant \tilde{K}_3 \Omega^{m+b} e(-2\pi m x)\, L_1(e(2\pi x))\, w(e(-2\pi x)) \text{ as } x \to \infty.$$

Therefore,

$$|f(e(2\pi x))| \leqslant K_3 \sup_{\delta e(2\pi x(\beta-1)) L_2(e(2\pi x)) \leqslant 8|v| \leqslant 4d} |f(e(2\pi(x-v)))\, e(-\xi\Omega^2 v^2/2)|$$
$$+ K_4 \Omega^{m+b} e(-2\pi m x)\, L_1(e(2\pi x))\, w(e(-2\pi x)).$$

Choosing $\Omega = \Omega_0 e(2\pi(1-\beta)\, x)/L_2(e(2\pi x))$ and Ω_0 sufficiently large, the conclusion of Theorem 1 follows by iteration.

3. Fourier Gap Series

We now formulate some corollaries to Theorem 1. Given

$$g(x)=\sum_{k=0}^{\infty}(a_k\cos\lambda_k x+b_k\sin\lambda_k x)=:c(x)+s(x), \tag{3.1}$$

where $\{\lambda_k\}$ increases $(\lambda_0=0)$ and $\lambda_{k+1}-\lambda_k\geqslant r_0L_2(\lambda_k)\,\lambda_K^\beta$, $r_0>0$, $k\geqslant k_0$, $0<\beta\leqslant 1$.

Corollary 1. *Suppose* $\sum_{k=0}^{\infty}(|\alpha_k|+|b_k|)<\infty$, *and* L_i, w *are given as in Theorem* 1. *Then the conditions*

$$\left.\begin{matrix}c(0)-c(x)\\ s(x)\end{matrix}\right\}=O(x^mL_1(1/x)\,w(x))\ [\text{resp. } o(\cdots)]\ \text{as } x\to 0+,\ \begin{cases}m\geqslant 0\\ m>0\end{cases}$$

imply that

$$\sum_{k=n}^{\infty}\begin{Bmatrix}a_k\\ b_k\end{Bmatrix}=O(\lambda_n^{-m\beta+b(1-\beta)}\,L_1(\lambda_n)\,L_2^{-m-b}(\lambda_n)\,w(1/\lambda_n))$$

[resp. $o(\cdots)$] *as* $n\to\infty$.

Proof. Put $f(u)=\sum_{k=0}^{\infty}a_k$, $u=0$, $f(u)=\sum_{k=n}^{\infty}\{\begin{smallmatrix}a_k\\ b_k\end{smallmatrix}\}$, $\lambda_{n-1}<u\leqslant\lambda_n$, $n=1, 2, 3,\ldots$. Then $c(x)$, $s(x)$ of (3.1) can be written as

$$c(x)=-\int_{0-}^{\infty}\cos(xu)\,df(u)=c(0)-x\int_0^{\infty}\sin(xu)\,f(u)\,du,$$

$$s(x)=-\int_0^{\infty}\sin(xu)\,df(u)\;\;=x\int_0^{\infty}\cos(xu)\,f(u)\,du.$$

Note that $\sqrt{\pi u/2}\;J_{-1/2}(u)=\cos u$ and $\sqrt{\pi u/2}\;J_{1/2}(u)=\sin u$ are Fourier kernels and satisfy the conditions $(C_1),\ldots,(C_4)$. Thus we can apply Theorem 1 directly.

Differentiability in the case $\beta=1$, $L_2(u)=1$, i.e., for Hadamard gaps, was completely treated by Freud [4, 5], Hsieh T'ing-Fan [10], Belov [1], and in [12]. This result is contained in:

Corollary 2. (a) *If* $g(x)$ of (3.1) *is differentiable in at least one point, then*

$$\lambda_n a_n\to 0,\ \lambda_n b_n\to 0\ \ as\ n\to\infty.$$

(b) *If*

$$\lambda_n(\alpha_n^2+b_n^2)^{1/2}\to 0\ \ as\ n\to\infty,$$

then $g(x)$ is smooth and differentiable in each interval at an infinite set of points. $g(x)$ is differentiable at those and only at those points x where the series obtained by formal differentiation is convergent and the derivative of g coincides with this sum.

Remark. Part (a) follows directly from Corollary 1; part (b) must be obtained independently.

Setting $b=0$, $L_1(u)=L_2(u)=1$, $w(x)=1$, our Corollary 1 contains another well known precise result given by Hsieh T'ing-Fan [10]:

COROLLARY 3. *If $g(x)$ satisfies a Lipschitz condition of order m, $m>0$, at x_0, then it follows that*

$$a_n, b_n = O(\lambda_n^{-m\beta}) \text{ as } n \to \infty.$$

If $g(x)$ is differentiable at x_0, then

$$a_n, b_n = o(\lambda_n^{-\beta}) \text{ as } n \to \infty.$$

Remark. Hardy [9] and Hsieh T'ing-Fan [10] establish slightly more: E.g., the series $\sum_{n=1}^{\infty} (\sin \pi n^2 x)/\pi n$ is divergent for certain irrational x and nowhere differentiable.

EXAMPLE. The function

$$g(x) = \sum_{n=1}^{\infty} \frac{\sin(e(n^{\nu})\, x + \phi_n)}{e(n^{\nu})\, n^{\mu-1}} \qquad (0<\nu\leqslant 1, \mu\leqslant\nu, \phi_n\in\mathbb{R})$$

is nowhere differentiable.

Here $m=\beta=1$ and $L_2(u)=\log^{(\nu-1)/\nu} u$, $w(x)=1$. We may assume without loss of generality that $g'(x_0)=0$. It follows that

$$n^{1-\mu}/e(n^{\nu}) = o(n^{1-\nu}/e(n^{\nu})) \text{ as } n\to\infty.$$

This contradiction establishes the result.

Note that Prohorenko [17] showed the nondifferentiability of this $g(x)$ for almost all real x where $\nu\geqslant 1/2$, $\mu\leqslant 3/2$.

4. FURTHER APPLICATIONS

Next, we study the special sequence $\{\lambda_k\}$, $\lambda_k = k^{\nu}\log^{\rho} k$, $k\geqslant k_0$, $\nu>1$, $\rho\in\mathbb{R}$. Then,

$$\lambda_{k+1}-\lambda_k \geqslant c_1 \lambda_k^{1-1/\nu} \log^{\rho/\nu} \lambda_k, \qquad c_1>0.$$

Put $L_2(u) = \log^{\rho/\nu} u$, $b=0$, $w(x)=1$, $\beta = 1 - 1/\nu$, $m > 0$. In this regard we obtain:

COROLLARY 4. *Under the hypothesis*

$$g(x) - g(x_0) = O(|x - x_0|^m L_1(1/|x - x_0|)) \text{ [resp. } o(\cdots)]$$

as $x \to x_0$ *it follows that*

$$\sigma_n(x_0) := \sum_{k=n}^{\infty} a_k \begin{Bmatrix} \cos \lambda_k x_0 \\ -\sin \lambda_k x_0 \end{Bmatrix} + b_k \begin{Bmatrix} \sin \lambda_k x_0 \\ \cos \lambda_k x_0 \end{Bmatrix}$$
$$= O(n^{(1-\nu)m} L_1(n) \log^{-\rho m} n) \text{ [resp. } o(\cdots)] \text{ as } n \to \infty.$$

Proof. By Corollary 1 one has

$$\sigma(x_0) = O(\lambda_n^{-m(1-1/\nu)} L_1(\lambda_n) \log^{-\rho m/\nu} \lambda_n) \text{ as } n \to \infty,$$
$$\lambda_n^{-m(1-1/\nu)} \log^{-\rho m/\nu} \lambda_n = O(n^{(1-\nu)m} \log^{\rho m(1-\nu)/\nu} n \log^{-\rho m/\nu} n).$$

EXAMPLE. The function

$$g(x) = \sum_{n=2}^{\infty} \frac{\sin(\pi x n^\nu \log^\rho n + \phi_n)}{\pi n^{\nu-1} \log^\rho n}$$

for $\nu > 2$ or $\rho > 1$ if $\nu = 2$ is continuous but nowhere differentiable. So series having all relevant types of gaps are covered by the various corollaries of the general theorem.

5. RIEMANN'S FUNCTION

We now focus our attention to the special gap series

$$g_\mu(x) = \sum_{n=1}^{\infty} \frac{\sin \pi n^2 x}{\pi n^\mu}, \qquad 1 \leqslant \mu < 3. \tag{5.1}$$

A recent informative report by Butzer and Stark [2] sheds new light on the fascinating history of this function. As is mentioned in the articles of Neuenschwander [14] and Segal [19], it was Hardy [8] who took up this example of a "nondifferentiable" function and proved that g_2 has no finite derivative at any point x, where x is either irrational or rational of the form $2A/(4B+1)$ or $(2A+1)/(2B)$, and more generally that g_μ is nondifferentiable at any irrational value of x if $\mu < 5/2$.

Hardy's proof depends substantially on results by himself and Littlewood in a paper on Diophantine Approximation published in 1914 [9]. They present a detailed investigation of special elliptic θ-functions and of the corresponding trigonometrical series. We recall a fundamental property of Gaussian sums [9, p. 195], namely

$$\begin{aligned} S_{2s}(r/s) = \sum_{k \bmod 2s} e^{i\pi k^2 r/s} &= (\pm 1 \pm i)\sqrt{2s}, && \frac{r}{s} = \frac{2A+1}{2B}\ (A, B \text{ integers}) \\ &= \pm 2\sqrt{s}, && r/s = 2A/(4B+1) \\ &= \pm 2i\sqrt{s}, && r/s = 2A/(4B+3) \\ &= 0, && \frac{r}{s} = \frac{2A+1}{2B+1}. \end{aligned} \tag{5.2}$$

This property is the key to the differentiability of g_μ.

Gerver [6, 7] in two lengthy papers, Queffelec [18] in his thesis, and Smith [20] in a short and elegant manner show that g_2 has no finite derivative at any rational point other than those of the form $x = (2A+1)/(2B+1)$, A, B integers. Later, Mohr [13] and Itatsu [11] published other nice proofs of this result.

Unfortunately, Smith's article contains some mistakes, and one of them seems to be rather serious. Indeed, the integral obtained by a transformation in the proof of Lemma 2 [20] diverges, and on p. 466 the relation (10) is to be replaced by

$$f(x \pm h^2) = f(x) - 2^{1/2}[S(x) \mp C(x)]h/s + O(h^3 s^{3/2}).$$

Therefore, the paragraph "Derivates at other points" (i.e., the irrational points) seems to be erroneous since

$$(x_n - x)^{1/2}(4q_n + 1) \to \infty \text{ as } n \to \infty.$$

More precisely, his $h \cdot s$ does not tend to zero and so neither do the remainder terms at irrational values of x which are poorly approximated by rational x_n.

We cannot handle the case where x is irrational by our methods either, because Corollary 4 applies only to rational points without further explanations. Thus, one must go back to Hardy's original proof, which utilizes elliptic θ-function theory. No other proof seems to be known up to this day. In this context Segal's remark (cf. [19, p. 81]): "In any case, no proof utilizing elliptic functions (or even indicating their relevance) is known" is somewhat surprising. We have a further

COROLLARY 5. *Let* $x_0 = r/s$, $s > 0$, $(r, s) = 1$, $1 < \mu \leqslant 2$. *If* $g_\mu(x)$ *of* (5.1) *is differentiable at the point* r/s, *then*

$$S_{2s}(r/s) = \sum_{k=1}^{2s} e(i\pi k^2 r/s) = 0.$$

Proof. Assume that $\sum_{k=1}^{2s} e(i\pi k^2 r/s) = a + ib \neq 0$, and $g'(x_0)$ exists. By Corollary 4 we conclude

$$\sum_{k=n}^{\infty} e(i\pi k^2 r/s)/k^\mu = o(1/n) \text{ as } n \to \infty.$$

But the tail sum has the asymptotic behavior

$$\sum_{k=2Ms+1}^{\infty} e(i\pi k^2 r/s)/k^\mu = \sum_{t=1}^{2s} \sum_{k=M}^{\infty} e(i\pi t^2 r/s)/(2ks+t)^\mu$$

$$= c_2(a+ib)/(2Ms+1)^{\mu-1} + O(1/(2Ms+1)^\mu) \text{ as } M \to \infty,\ c_2 > 0.$$

This leads to a contradiction. Thus Tauberian theorems yield necessary conditions for differentiability.

In the same way we can generalize the Tauberian implication of a theorem due to Queffelec [18]. We consider a real function P, where P is positive, nondecreasing and

$$P(u) \sim c_3 u^\nu \text{ if } u \geqslant u_0, \qquad \nu > 1,$$

for which further there exists a positive integer t with

$$P(n+t)\, x_0 \equiv P(n)\, x_0 (\text{mod } 2), \qquad n \geqslant n_0.$$

Indeed, if the series

$$g_{P,\kappa}(x) := \sum_{n=1}^{\infty} \frac{\sin(\pi P(n)\, x)}{\pi P(n)^\kappa}, \qquad \kappa \leqslant 1$$

is differentiable at x_0, then

$$\sum_{n=1}^{2t} e(i\pi P(n)\, x_0) = 0. \tag{5.3}$$

The Abelian assertion, namely that (5.3) is also sufficient for the differentiability of $g_{P,1}$ at the point x_0, was announced by Gerver [7] and Queffelec [18]. The relevance of (generalized) Gaussian sums in all proofs, whether of Tauberian or Abelian type, is quite evident.

6. Riemann's Function, Continued

In this paragraph we want to study the degree of differentiability of the example

$$g(x)=\sum_{n=1}^{\infty}\frac{\sin \pi n^2 x}{\pi n^{\mu}}, \qquad 1<\mu<3.$$

By generalizing the method of Smith [20] we will show that

$$\sum_{n=1}^{\infty}\frac{\sin \pi n^{\nu} x}{\pi n^{\nu-1/2}}$$

has no finite derivative at any rational $x_0=r/s$. Therefore,

$$g_{3/2}(x)=\sum_{n=1}^{\infty}\frac{\sin \pi n^2 x}{\pi n^{3/2}}$$

is nowhere differentiable by taking into account Hardy's result for the irrational x_0. We will use the Smith technique because we are unable to prove or disprove the following assertions:

Define

$$f_{\mu}(x)=\sum_{n=1}^{\infty}\frac{\sin(\pi n^{\nu}x+\phi_n)}{\pi n^{\mu}}, \qquad \mu, \nu>1.$$

Then the (non)differentiability of f_{μ} at the point x implies the (non)existence of $f'_{\mu_1}(x)$, $\mu_1>\mu$ $(\mu_1<\mu)$.

Several lemmas will be needed. For the first see Oberhettinger [15, p. 23, 136].

Lemma 1. *For* $-1<\mu<3$, $\mu\neq 1$ *one has*

$$\int_0^{\infty}\frac{\sin \pi x^2}{x^{\mu}}\cos(2\pi xy)\,dx$$

$$=-\left(\frac{\pi}{2}\right)^{(\mu-1)/2}\Gamma\left(\frac{1-\mu}{2}\right)\operatorname{Im}\left\{e\left(-\frac{\pi}{4}(1-\mu)\,i\right){}_1F_1\left(\frac{1-\mu}{2};\frac{1}{2};i\pi y^2\right)\right\},$$

$$\int_0^{\infty}\frac{\sin \pi x^2}{x^{\mu}}\sin(2\pi xy)\,dx$$

$$=-y\pi^{\mu/2}\Gamma\left(\frac{2-\mu}{2}\right)\operatorname{Im}\left\{e\left(-\frac{\pi}{4}(2+\mu)\,i\right){}_1F_1\left(\frac{2-\mu}{2};\frac{3}{2};-i\pi y^2\right)\right\}.$$

LEMMA 2. *For $|y| \to \infty$ one has*

$$\int_0^\infty \frac{\sin \pi x^2}{x^\mu} \cos(2\pi xy)\, dx$$

$$= \frac{\mu-2}{4} \Gamma\left(\frac{3-\mu}{2}\right) \pi^{\mu-3/2} |y|^{\mu-3} / \Gamma\left(\frac{\mu}{2}\right)$$

$$-\frac{1}{2} |y|^{-\mu} \sin\left(\pi\left(y^2 - \frac{1}{4}\right)\right) + O(|y|^{\mu-4} + |y|^{-\mu-1}),$$

$$\int_0^\infty \frac{\sin \pi x^2}{x^\mu} \sin(2\pi xy)\, dx$$

$$= \frac{\mu-1}{4} \Gamma\left(\frac{4-\mu}{2}\right) \pi^{\mu-3/2} \operatorname{sgn} y |y|^{\mu-3} / \Gamma\left(\frac{1+\mu}{2}\right)$$

$$+\frac{1}{2} \operatorname{sgn} y |y|^{-\mu} \sin\left(\pi\left(y^2 + \frac{1}{4}\right)\right) + O(|y|^{\mu-4} + |y|^{-\mu-1}).$$

Proof. For fixed values a, c and as $|z| \to \infty$ (see [16, p. 256])

$$_1F_1(a; c; z) = \frac{\Gamma(a)}{\Gamma(c-a)} e(\pm i\pi a)\, z^{-a}(1 + a(c-a-1)/z + O(1/|z|^2))$$

$$+ \frac{\Gamma(c)}{\Gamma(a)} e(z)\, z^{a-c}(1 + (1-a)(c-a)/z + O(1/|z|^2)),$$

where the sign is taken as positive or negative according as $\operatorname{Im} z > 0$ or $\operatorname{Im} z < 0$, $-\pi < \arg z < \pi$. The conclusion of Lemma 2 follows by the aid of Lemma 1.

An elementary but tedious proof can also be given by the method of the stationary phase as in Olver [16, p. 96–104]. Indeed, without loss of generality we may assume $0 < \mu < 1$. Then consider the integrals

$$I_\pm = \int_0^\infty \frac{e(i\pi(u^2 \pm 2uy))}{u^\mu}\, du \text{ as } y \to \infty.$$

Substituting $t = (u/y)^{1-\mu}$, $\pi y^2 = x$, $1/(1-\mu) = \alpha$, yields

$$I_\pm = \alpha y^{1/\alpha} \int_0^\infty e(\mathrm{ix} p(t))\, dt,\ p(t) := t^{2\alpha} \pm 2t^\alpha.$$

To evaluate I_+ we apply Theorem 13.1 [16] twice and obtain two terms of the asymptotic expansion (as $y \to \infty$) in the form

$$I_+ = e(i\pi(1-\mu)/2)\, \Gamma(1-\mu)(2\pi y)^{\mu-1}\, (1 + (2-\mu)(1-\mu)/(4i\pi y^2) + \cdots).$$

It is more complicated to estimate I_- since $p'(t)$ vanishes at $t=0$ and $t=1$. We split I_- into three integrals and find

$$I_- = e(-i\pi(1-\mu)/2)\,\Gamma(1-\mu)(2\pi y)^{\mu-1}(1-(2-\mu)(1-\mu)/(4i\pi y^2)+\cdots)$$
$$+ y^{-\mu}e(i\pi(\tfrac{1}{4}-y^2))+\cdots, \text{ as } y\to\infty.$$

LEMMA 3. *Define*

$$\phi_1=\frac{\sin\pi x}{\pi|x|^{\mu/2}},\ x\neq 0,\qquad \phi_2=\frac{1-\cos\pi x}{\pi|x|^{\mu/2}},\qquad x\neq 0,\ \phi_2(0)=0,\ 1<\mu<3.$$

Then the Fourier transform h of $\phi_1(x^2)+i\phi_2(x^2)$ *has the expansion for* $|y|\to\infty(\psi_j(x):=\phi_j(x^2),\ j=1,2)$

$$h(y)=\hat{\psi_1}(y)+i\hat{\psi_2}(y):=\int_{-\infty}^{\infty}\frac{e(i\pi x^2)-1}{i\pi|x|^\mu}e(-2\pi ixy)\,dx$$
$$=\frac{e(-i\pi(y^2-1/4))}{i\pi|y|^\mu}+\frac{\mu-2}{2}\Gamma\left(\frac{3-\mu}{2}\right)\pi^{\mu-5/2}|y|^{\mu-3}/\Gamma\left(\frac{\mu}{2}\right)$$
$$+O(y^{-2}+|y|^{\mu-4}).$$
$$h(0)=\frac{1}{i}\Gamma\left(\frac{1-\mu}{2}\right)\pi^{(\mu-3)/2}e(i\pi(1-\mu)/4).$$

Proof. Using Lemma 2,

$$\int_{-\infty}^{\infty}\frac{e(i\pi x^2)-1}{i\pi|x|^\mu}(\cos(2\pi xy)-i\sin(2\pi xy))\,dx$$
$$=\frac{2}{i\pi}\int_0^\infty\frac{\cos(\pi x^2)-1}{x^\mu}\cos(2\pi xy)\,dx$$
$$+\frac{2}{\pi}\int_0^\infty\frac{\sin\pi x^2}{x^\mu}\cos(2\pi xy)\,dx.$$

Integrating the first term on the right by parts gives for $y\neq 0$

$$\int_0^\infty\frac{\cos(\pi x^2)-1}{x^\mu}\cos(2\pi xy)\,dx$$
$$=\frac{\mu}{2\pi y}\int_0^\infty\frac{\cos(\pi x^2)-1}{x^{\mu+1}}\sin(2\pi xy)\,dx+\frac{1}{y}\int_0^\infty\frac{\sin\pi x^2}{x^{\mu-1}}\sin(2\pi xy)\,dx$$
$$=\frac{1}{4}\pi^{\mu-5/2}(\mu-2)\,\Gamma\left(\frac{5-\mu}{2}\right)|y|^{\mu-5}/\Gamma\left(\frac{\mu}{2}\right)$$
$$+\tfrac{1}{2}|y|^{-\mu}\cos(\pi(y^2-1/4))+O(y^{-2}+|y|^{\mu-6})\text{ as }|y|\to\infty.$$

The result for $\mu=2$ remains valid without the second term and can be obtained in an elementary way.

Now we shall make use of Lemma 1 in [20] to obtain an estimate of

$$Q_i(\alpha) := \sum_{k=-\infty}^{\infty} h\psi_i(hk+h\alpha) \text{ as } h\to 0+, \qquad i=1,2. \tag{6.1}$$

It will depend on the Poisson summation formula. Here we need (see [20]):

LEMMA 4. *If* $\psi \in C(\mathbb{R})$, $\psi(x)=O(|x|^{-\mu})$ *as* $|x|\to\infty$, $\mu>1$, *and* $|\hat{\psi}(y)|\,|y|^{\beta} \leqslant K$, $\beta>1$, *it follows that* $Q(\alpha)=\hat{\psi}(0)+O(h^{\beta})$ *as* $h\to 0+$ *for any real* α.

LEMMA 5. *There hold the estimates*

$$\begin{aligned}\sum_{\substack{k=-\infty\\(hk+h\alpha\neq 0)}}^{\infty} &(h\psi_1(hk+h\alpha)+ih\psi_2(hk+h\alpha))\\ &= h(0)+c_4(\alpha)\,h^{3-\mu}+\frac{2}{i}h^{\mu}\sum_{k=1}^{\infty}\frac{\cos 2\pi k\alpha}{\pi k^{\mu}}e\left(i\pi\left(\frac{1}{4}-\left(\frac{k}{h}\right)^2\right)\right)\\ &\quad + O(h^2+h^{4-\mu}) \qquad (1<\mu<3,\ c_4(\alpha)\in\mathbb{R};\ h\to 0+).\end{aligned}$$

Proof. In the case $1<\mu\leqslant 2$ Lemma 4 is directly applicable, in the case $2<\mu<3$ only to the function ψ_2.

To estimate $\sum_{k=-\infty}^{\infty} h\psi_1(hk+h\alpha)$ we introduce the function $H(x) := |x|^{2-\mu}/(1+x^2)$ having the same asymptotic behavior as ψ_1 in the neighborhood of the origin. We find (see [15, p. 5])

$$\begin{aligned} I &= \int_0^{\infty} x^{2-\mu}\frac{\cos 2\pi xy}{1+x^2}\,dx\\ &= \int_0^{\infty} x^{2-\mu}\cos 2\pi xy\,dx - \int_0^{\infty} x^{4-\mu}\frac{\cos 2\pi xy}{1+x^2}\,dx =: I_1 - I_2,\\ I_1 &= \frac{\mu-2}{4}\pi^{\mu-5/2}\Gamma\left(\frac{3-\mu}{2}\right)|y|^{\mu-3}/\Gamma\left(\frac{\mu}{2}\right).\end{aligned}$$

Integrating by parts we obtain

$$\begin{aligned} I_2 &= \frac{1}{2\pi y}\left(-(4-\mu)\int_0^{\infty} x^{3-\mu}\frac{\sin 2\pi xy}{1+x^2}\,dx + 2\int_0^{\infty} x^{5-\mu}\frac{\sin 2\pi xy}{(1+x^2)^2}\,dx\right)\\ &= \frac{1}{4\pi^2 y^2}\left(-(4-\mu)(3-\mu)\int_0^{\infty} x^{2-\mu}\frac{\cos 2\pi xy}{1+x^2}\,dx + O(y^{-2})\right).\end{aligned}$$

These estimates yield

$$I=I_1(1+(4-\mu)(3-\mu)/(4\pi^2 y^2)+O(y^{-3})) \text{ as } |y|\to\infty.$$

Now we can apply Lemma 4 to $\psi_1 - H$, but we must first find the asymptotic expansion of $\sum_{k=-\infty}^{\infty} hH(hk+h\alpha)$, $0<\alpha<1$. For this purpose we proceed as in [3, pp. 153–163]. Choose a contour $C_{N,\theta}$ symmetric about the real axis, consisting of a segment of the line $Re\, t=N+1/2+\alpha$ and two curves $C_{N,\theta}^+$ and $C_{N,\theta}^-$. The curve $C_{N,\theta}^+$ is a part of a curve C_θ^+, going from the point $t=\theta$ to infinity in the upper half-plane, and meeting each line $Re\, t=A$, $A>\theta$, exactly once. The curve $C_{N,\theta}^-$ is symmetric to $C_{N,\theta}^+$ in the real axis.

Consider now the integral

$$I_{N,\theta}=\frac{1}{2i}\oint_{C_{N,\theta}} t^{2-\mu}\frac{\cot \pi(t-\alpha)}{z^2+t^2}\,dt, \qquad 0<\theta<\alpha.$$

Passing to the limit as $N\to\infty$, we obtain

$$\begin{aligned} I_\theta &= \sum_{k=0}^{\infty} (\alpha+k)^{2-\mu}/(z^2+(k+\alpha)^2)=\int_\theta^\infty t^{2-\mu}/(t^2+z^2)\,dt \\ &\quad +\int_{C_\theta^+} t^{2-\mu}/[(z^2+t^2)(e(-2\pi i(t-\alpha))-1)]\,dt \\ &\quad +\int_{C_\theta^-} t^{2-\mu}/[(z^2+t^2)(e(2\pi i(t-\alpha))-1)]\,dt \\ &=\int_\theta^\infty t^{2-\mu}/(z^2+t^2)\,dt+d_0(\theta)/z^2-d_1(\theta)/z^4+-\cdots \end{aligned} \tag{6.2}$$

where

$$d_k(\theta)=\int_{C_\theta^+}\frac{t^{2k+2-\mu}}{e(-2\pi i(t-\alpha))-1}\,dt+\int_{C_\theta^-}\frac{t^{2k+2-\mu}}{e(2\pi i(t-\alpha))-1}\,dt,$$

if we expand the sum of the last two integrals in (6.2) in an asymptotic series in powers of $1/z^2$ and integrate term by term. We take $\theta=0$ and evaluate the principal term. Thus,

$$\sum_{k=-\infty}^{\infty} hH(hk+h\alpha)=\pi/\cos\left(\frac{2-\mu}{2}\pi\right)+c_5(\alpha)\,h^{3-\mu}+O(h^{5-\mu}),$$

and similarly

$$\sum_{k=1}^{\infty} hH(hk) = \left(\frac{\pi}{2}\right) \Big/ \cos\left(\frac{2-\mu}{2}\pi\right) + c_5(0)\, h^{3-\mu} + O(h^{5-\mu}) \text{ as } h \to 0+.$$

This will finally lead to the proof of our second main theorem:

Proof of Theorem 2. For $\mu = 1$ and $\mu = 2$ the result is given by Hsieh T'ing-Fan [10] and Smith [20]. In the other cases we proceed as in [20]. Indeed,

$$\begin{aligned}(g(x+h^2) + g(x-h^2))/2 &= \sum_{n=1}^{\infty} \frac{\sin \pi n^2 x}{\pi n^\mu} \cos \pi n^2 h^2 \\ &= g(x) - \frac{h^\mu}{2} \sum_{n=-\infty}^{\infty} \sin(\pi n^2 x)\, \psi_2(nh).\end{aligned}$$

Writing $n = 2ks + t$, $0 \leqslant t \leqslant 2s-1$, then $\sin(\pi(2ks+t)^2 r/s) = \sin(\pi t^2 r/s)$, and by Lemma 5

$$\begin{aligned}&(g(x+h^2) + g(x-h^2))/2 \\ &= g(x) - \frac{h^{\mu-1}}{4s} \sum_{t=0}^{2s-1} \sin \pi t^2 x \sum_{n=-\infty}^{\infty} 2sh\psi_2(2ksh + th) \\ &= g(x) - \frac{h^{\mu-1}}{4s} \sum_{t=0}^{2s-1} \sin(\pi t^2 r/s) \cdot \{\hat{\psi_2}(0) \\ &\quad - 2(2sh)^\mu \sum_{k=1}^{\infty} \frac{\cos(\pi k t/s)}{\pi k^\mu} \cos\left(\pi\left(\left(\frac{k}{2sh}\right)^2 - \frac{1}{4}\right)\right) \\ &\quad + O((sh)^2 + (sh)^{4-\mu})\}.\end{aligned}$$

Similarly, as $h \to 0+$,

$$\begin{aligned}&(g(x+h^2) - g(x-h^2))/2 \\ &= \sum_{n=1}^{\infty} \frac{\cos \pi n^2 x}{\pi n^\mu} \sin \pi n^2 h^2 \\ &= \frac{h^{\mu-1}}{4s} \sum_{t=0}^{2s-1} \cos(\pi t^2 r/s) \sum_{\substack{k=-\infty \\ 2ks+t \neq 0}}^{\infty} 2sh\psi_1(2skh + th) \\ &= \frac{h^{\mu-1}}{4s} \sum_{t=0}^{2s-1} \cos(\pi t^2 r/s) \cdot \{\hat{\psi_1}(0) + c_4(t)(2hs)^{3-\mu} \\ &\quad + 2(2sh)^\mu \sum_{k=1}^{\infty} \frac{\cos(\pi k t/s)}{\pi k^\mu} \sin\left(\pi\left(\frac{1}{4} - \left(\frac{k}{2sh}\right)^2\right)\right) \\ &\quad + O((sh)^2 + (sh)^{4-\mu})\}.\end{aligned}$$

But $\sum_{t=0}^{2s-1} e(i\pi t^2 r/s) = 0$ if and only if $rs \equiv 1 \pmod 2$, (cf. (5.1)). Thus, for $3/2 < \mu < 3$,

$$\lim_{h^2 \to 0} (g(x+h^2) + g(x-h^2) - 2g(x))/h^2 = 0,$$

and the symmetric derivative $\lim_{h^2 \to 0}(g(x+h^2) - g(x-h^2))/h^2$ exists if and only if $x = (2A+1)/(2B+1)$, A, B integers.

To treat the case $3/2 \geqslant \mu > 1$, $rs \equiv 1 \pmod 2$, consider

$$\chi(h) := \sum_{t=0}^{2s-1} \cos(\pi t^2 r/s) \sum_{k=1}^{\infty} \frac{\cos(\pi k t/s)}{k^\mu} \sin\left(\pi\left(\left(\frac{k}{2sh}\right)^2 - \frac{1}{4}\right)\right)$$

$$= \sum_{k=1}^{\infty} (d_k/k^\mu) \sin\left(\pi\left(\left(\frac{k}{2sh}\right)^2 - \frac{1}{4}\right)\right),$$

$$d_k := \sum_{t=0}^{2s-1} \cos \pi t^2 \frac{r}{s} \cos \pi k \frac{t}{s}.$$

It is easy to verify that

$$d_k = 0 \qquad \text{for } k \text{ even}, \qquad d_k = d_{2s \pm k}, \qquad \sum_{k=1}^{2s} d_k = 2s.$$

We put $(2sh)^{-2} = u$. Then $\sum_{k=1}^{N} (d_k/k^\mu) \sin(\pi(k^2 u - \frac{1}{4}))$ converges absolutely and uniformly as $N \to \infty$ to the continuous, nonconstant, 2-periodic function $\tilde{\chi}(u)$. Thus, using the uniqueness theorem for Fourier series, $\lim_{u \to \infty} \tilde{\chi}(u)$ does not exist, and therefore the same holds for the symmetric derivative at $x = r/s$. For the other points see Corollary 5.

Similar arguments hold for the almost periodic function

$$\sum_{k=1}^{\infty} (d_k/k^\mu) \sin(\pi(\lambda_k u - \lambda_0))$$

which occurs in the following section.

7. The Case of More General Gaps

Finally, we want to give some hints on how to handle the case

$$g(x) = \sum_{n=1}^{\infty} \frac{\sin \pi n^\nu x}{\pi n^{\nu - 1/2}}, \qquad \nu \geqslant 2,\ \nu \in \mathbb{N}.$$

To carry over the method of Hardy (and Littlewood) [8, 9] and to show that $g(x)$ has no finite derivative for any irrational x seems to be very difficult. To the best of our knowledge no further results are known.

However, the nondifferentiability at rational values of x can be settled as in Section 6. Indeed, to evaluate

$$(g(x+h^\nu)+g(x-h^\nu)-2g(x))/h^\nu,$$

we must study the asymptotic behavior of

$$\int_0^\infty \frac{\cos(\pi x^\nu)-1}{\pi x^{\nu-1/2}} \cos(2\pi xy)\,dx \text{ as } y \to \infty.$$

This leads after integrating by parts to the integral

$$\int_0^\infty \frac{\sin \pi x^\nu}{\pi x^{1/2}} \sin(2\pi xy)\,dx.$$

If we apply the saddlepoint method to $H(y) := \int_0^\infty x^\gamma e(-x^\nu)\, e(xy)\,dx$ as in [3, p. 141], we find the crucial asymptotic estimate

$$\int_0^\infty \frac{\cos(\pi x^\nu)-1}{\pi x^{\nu-1/2}} \cos(2\pi xy)\,dx \sim c_6 y^{-3/2} \sin(\pi c_7(y^{\nu/(\nu-1)}-c_8)) \text{ as } y \to \infty.$$

To complete the argument one proceeds as in the proof of Theorem 2.

ACKNOWLEDGMENTS

The author is indebted to Professor P. L. Butzer for detailed and valuable comments and to Professor E. Stark for his criticisms of an earlier draft and many helpful suggestions.

REFERENCES

1. A. S. BELOV, (A) study of some (certain) trigonometric series, *Math. Notes* **13** (1973), 291–298, translated from *Mat. Zametki* **13** (1973), 481–492.
2. P. L. BUTZER AND E. L. STARK, "Riemann's Example" of a Continuous Nondifferentiable Function in the Light of Two Letters (1865) of Christoffel to Prym, to appear.
3. M. A. EVGRAFOV, "Asymptotic Estimates and Entire Functions," Gordon & Breach, New York, 1961.
4. G. FREUD, Über trigonometrische Approximation und Fouriersche Reihen, *Math. Z.* **78** (1962), 252–262.
5. G. FREUD, On Fourier series with Hadamard gaps, *Studia Sci. Math. Hungar.* **1** (1966), 87–96.
6. J. GERVER, The differentiability of the Riemann function at certain rational multiples of π, *Amer. J. Math.* **92** (1970), 33–55.

7. J. GERVER, More on the differentiability of the Riemann function, *Amer. J. Math.* **93** (1971), 33–41.
8. G. H. HARDY, Weierstraß's non-differentiable function, *Trans. Amer. Math. Soc.* **17** (1916), 301–325.
9. G. H. HARDY AND J. E. LITTLEWOOD, Some problems of diophantine approximation (II), *Acta Math.* **37** (1914), 193–238.
10. HSIEH T'ING-FAN, On lacunary Fourier series I, II, *Chinese Math. Acta* **5** (1964), 340–345; **8** (1966), 534–537.
11. S. ITATSU, Differentiability of Riemann's function, *Proc. Japan Acad. Math. Sci. Ser. A* **57** (1981), 492–495.
12. W. LUTHER, Taubersche Restgliedsätze für eine Klasse von Fourierkernen, *Mittl. Math. Sem. Gießen. Heft* **143** (1980).
13. E. MOHR, Wo ist die Riemannsche Funktion $\sum_{n=1}^{\infty} (\sin n^2x)/n^2$ nichtdifferenzierbar? *Ann. Math. Pura Appl.* (4) **123** (1980), 93–104.
14. E. NEUENSCHWANDER, Riemann's example of a continuous "nondifferentiable" function, *Math. Intelligencer* **1** (1978), 40–44.
15. F. OBERHETTINGER, "Tabellen zur Fourier-Transformation," Springer-Verlag, Berlin, 1957.
16. F. W. J. OLVER, "Asymptotics and Special Functions," Academic Press, New York, 1974.
17. V. I. PROHORENKO, Fourier series with lacunae, *Siber. Math. J.* **18** (1977), 272–283.
18. H. QUEFFELEC, Dérivabilité de certaines sommes de séries de Fourier lacunaires, *C. R. Acad. Sci. Paris* Sér. A **273** (1971), 291–293. Proofs in: Thèse. Université Paris Sud, 1971.
19. S. L. SEGAL, Riemann's example of a continuous "nondifferentiable" function continued, *Math. Intelligencer* **1** (1978), 81–82.
20. A. SMITH. The differentiability of Riemann's function, *Proc. Amer. Math. Soc.* **34** (1972), 463–468.